Propagation and attenuation of EM waves in complex media

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Abstract

We review the physical mechanisms behind extinction/absorption of electromagnetic waves in a medium. First the physical picture is explained without formulae, then a formal treatment is given of the macroscopic Maxwell equations coupled to various constitutive models. We examine the Debeye constitutive model in some detail and derive a formula for the skindepth within this model. We also derive the frequency dependent permittibity, and demonstrate negligible dispersion and deep penetration after fitting the model parameters to field data.

1 Physical ideas

Maxwell's theory

The famous Maxwell equations describe (amongst other things) the propagation of electromagnetic waves and their interaction with material objects. Two forms of the Maxwell equations are often confused but should be distinguished: Maxwell's equations in vacuum and Maxwell's equations in a medium.

As far as we know Maxwell's equations in a vacuum describe all electromagnetic phenomena in nature at the macroscopic level and, when extended to the quantum domain, at every scale that we have probed. This form of the theory gives equations that govern the electric and magnetic fields (**E** and **B**) and their interaction with charged or magnetic material objects in space-time. The theory has two fundamental constants of nature, the vacuum permittivity ϵ_0 and the vacuum permeability μ_0. Waves propagate freely with speed $c = 1/\sqrt{\epsilon_0 \mu_0}$.

The Maxwell equations in a medium are in essence the vacuum equations coupled to some very specific material objects, namely a classical continuum model of matter. In this case several additional fields are introduced such as the electric displacement field **D** and the magnetization field **H**. These are not fundamental fields but provide a convenient description of the medium such that the equations in a medium have a similar form as the equations in vacuum, with the fundamental constants ϵ_0 and μ_0 now becoming functions $\epsilon(\mathbf{x}, \omega)$ and $\mu(\mathbf{x}, \omega)$ of space and frequency. Except in the simplest cases these functions must be measured for specific materials.

A physical model that describes the measured $\epsilon(\mathbf{x}, \omega)$ and $\mu(\mathbf{x}, \omega)$ is called a constitutive model. Such models invariably describe the interaction of charged and dipole structures in the medium with the electromagnetic field. For a textbook on the subject see for example [16].

Attenuation in a medium

The physical picture of propagation and attenuation of a wave in a medium is that the wave sets structures in the medium such as dipoles in motion. These material dipoles have their own electromagnetic field which is then added to the incoming wave causing a phase shift, and hence what appears as a slower propagation speed. Moving these dipoles requires energy and this energy is taken out of the electromagnetic wave. The first conductivity models of this nature that were proposed are the Drude model [7, 8] and the Lorentz-Drude model [12, 13, 14]. A similar model for dielectrics was proposed by Debye [4]. These models are still thought to be essentially correct and have been extended and modified. We mention the widely used Cole-Cole model [2, 3] and the Debye model extended with inertial effects [15]. Though these physical models were originally thought to describe atomic and molecular structures, it was later realized that these models are quite universal and dipole oscillations can occur also in larger structures called “clusters” in the Dissado-Hill theory [5]. All these models are founded on the Debye model in that they consider a large population of various elementary Debye dipoles in a medium. This mechanism is generally called relaxation, for a review see [10]

If we assume the medium is homogeneous *and the wave is stationary (periodic)* this mechanism operates in the same fashion everywhere in the medium and it follows that the wave loses a constant fraction of its energy and amplitude per meter leading to an exponential decay in the direction of travel with a decay constant which is expressed as the “attenuation length” (sometimes called “skindepth”) L which is the length of travel after which the amplitude decays by a factor $1/e$. We can calculate L from theory if we are given a constitutive model. This is illustrated in Fig. 1 where we depict a monochromatic continuous wave that is emitted at $x = 0$ and propagates to the right in a lossy medium. The dipoles will oscillate with an amplitude proportional to the wave amplitude and the friction loss is proportional to the dipole amplitude, hence an exponential loss.

If the wave is not stationary, there may be significant transient effects before the dipoles start their regular vibration pattern and the above analysis breaks down. As it takes a few “pushes” from the electric field to get the dipoles to oscillate and the loss is proportional to the amplitude of the dipole oscillation the head of the wave

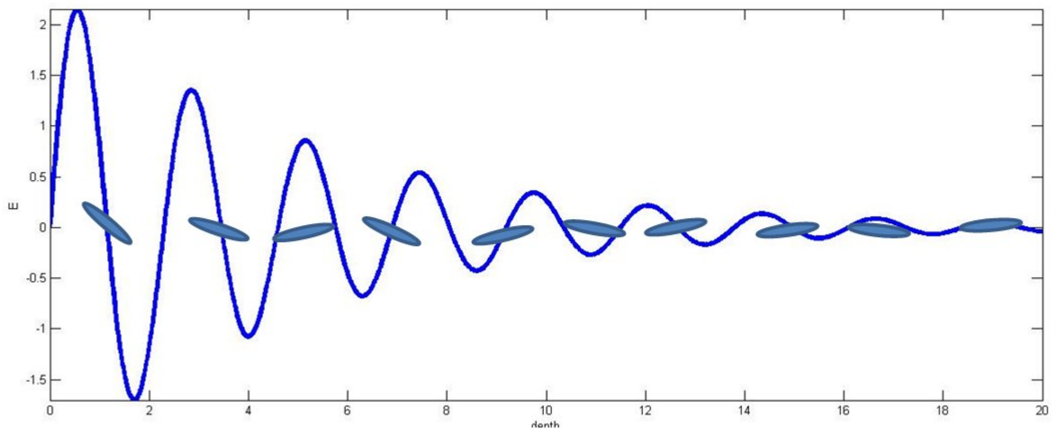

Figure 1: A monochromatic continuous wave is emitted at $x = 0$ and propagates to the right in a lossy medium. The perturbed dipoles are indicated.

suffers less loss than the following part. This is illustrated in Fig. 2. The first period

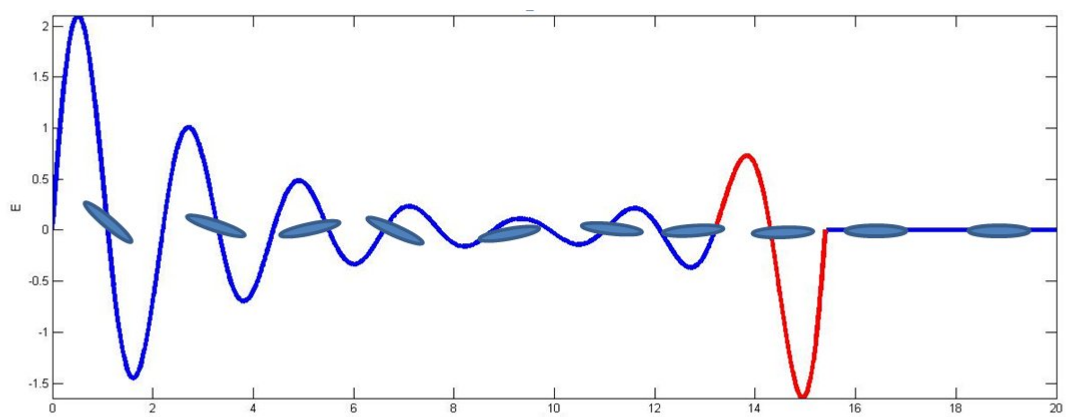

Figure 2: The same wave as in Fig. 1 right after the transmitter is turned on. The wave has propagated only as far as $x = 15$.

of the wave (drawn in red) encounters dipoles that are at rest, whereas the rest of the wave encounters dipoles that are already oscillation. Hence the head of the wave is less attenuated that the rest. For pulsed GPR the antenna is turned off after one period and the red part of the wave is all that is propagating.

How large this effect is cannot be determined from first principles. In the classical Debye model the effect amounts to about a 20% reduction in absorption, however the Debye model ignores the dipole mass, which plays a crucial role here in determining how long it takes before the dipole reaches maximum amplitude. When a mass term is included [15] the effect can possibly be quite large.

How large can only be determined experimentally by comparing attenuation of a pulse with attenuation of a continuous wave. However I speculate that the effect is large enough to contribute significantly to the unexpectedly large penetration depths achieved by Adrok.

2 Technical Details

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \cdot \mathbf{B} = 0 \tag{1a}$$

$$\nabla \times \mathbf{E} = -\mathbf{B}_t \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \mathbf{D}_t \tag{1b}$$

with $\mathbf{E}$ the electrical field, $\mathbf{B}$ the magnetic field, $\mathbf{D}$ the electric displacement field, $\mathbf{H}$ the magnetizing field, ρ_f the free charge density, $\mathbf{J}_f$ the free current density, and $_t$ the time derivative. We have the definitions

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} \tag{1c}$$

where $\mathbf{P}$ is the polarization and $\mathbf{M}$ the magnetization. $\epsilon_0 \approx 8.85\ 10^{-12}$, and $\mu_0\epsilon_0 = 1/c^2$ with $c \approx 3\ 10^8$ (SI units). $\mathbf{P}$ and $\mathbf{M}$ have to be specified by a specific material (or "constitutive") model.

In the frequency domain at angular frequency ω for isotropic linear materials we can write (it should be clear from the context when fields are time or frequency dependent)

$$\mathbf{P} = \epsilon_0 \chi(\omega) \mathbf{E} \qquad \mathbf{M} = \chi_m(\omega) \mathbf{B}/\mu_0 \tag{2}$$

with $\chi_{(m)}$ the electric (magnetic) susceptibility which results in

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 (1 + \chi) \mathbf{E} \qquad \mathbf{H} = \mathbf{B}/\mu = \mathbf{B}/(\mu_0(1 + \chi_m)) \tag{3}$$

with

$$\epsilon = \epsilon_0 (1 + \chi) = \epsilon_0 \epsilon_r \qquad \mu = \mu_0 (1 + \chi_m) = \mu_0 \mu_r \tag{4}$$

with $\epsilon_{(r)}$ the (relative) permittivity, and $\mu_{(r)}$ the (relative) permeability. Finally we write the conductivity model as

$$\mathbf{J}_f = \sigma(\omega) \mathbf{E} + \mathbf{J} \tag{5}$$

with σ the conductivity and $\mathbf{J}$ external currents not due to material conductivity. The whole zoo of frequency dependent "constants" are complex numbers unless they have subscript $_0$.

3 Pure electrics in an uncharged medium

Let's assume $\mathbf{M} = 0$ and $\rho_f = 0$ here. We can now eliminate the magnetic field and obtain (for divergence free $\mathbf{E}$)

$$\Delta\mathbf{E} = \mu_0\mathbf{J}^f_t + \mu_0\mathbf{D}_{tt}. \tag{6}$$

with Δ the Laplacian. In the frequency domain we get

$$\Delta\mathbf{E} + \mu_0(\epsilon\omega^2 - i\sigma\omega)\mathbf{E} = i\mu_0\omega\mathbf{J}. \tag{7}$$

Now let's find a plane wave solution of (7) without source current and compute the skindepth.

Consider a wave of the form

$$\mathbf{E} = \mathbf{e}_x e^{i(\omega t - kx)} \tag{8}$$

and we get the dispersion relation

$$k^2 = \mu_0(\epsilon\omega^2 - i\sigma\omega). \tag{9}$$

Given the redundant zoo of "constants" let's decide to express everything in terms of two complex frequency dependent quantities $\chi = \chi' - i\chi''$ and $\sigma = \sigma' + i\sigma''$ giving.

$$k^2 = \frac{\omega^2}{c^2}(1 + \chi' + \sigma''/(\omega\epsilon_0) - i(\chi'' + \sigma'/(\omega\epsilon_0)). \tag{10}$$

I have seen the definition here of a "permittivity" ρ

$$\rho = 1 + \chi' + \sigma''/(\omega\epsilon_0) - i(\chi'' + \sigma'/(\omega\epsilon_0)) \tag{11}$$

which name makes no sense, as the terms with σ have nothing to do with the polarization. It's still a useful variable, so I'll call it $\rho = \rho' - i\rho''$ instead. Experimentally it is often not possible to distinguish the effects of the conductivity and the polarization in which case a single complex constant ρ suffices. For our purposes we need to distinguish the polarization and the conductivity. Solving (10) gives for the wave number k

$$k = \frac{\omega}{c}\sqrt{\rho'/2}(\sqrt{\sqrt{\rho''^2/\rho'^2 + 1} + 1} + i\sqrt{\sqrt{\rho''^2/\rho'^2 + 1} - 1}). \tag{12}$$

Substituting (12) in (8) gives us the attenuation length

$$L = \frac{c}{\omega}/(\sqrt{\rho'/2}\sqrt{\sqrt{\rho''^2/\rho'^2 + 1} - 1}). \tag{13}$$

This formula agrees with for example [11, 17] For small losses this expression is well approximated by

$$L = \frac{2c\sqrt{\rho'}}{\omega\rho''}. \tag{14}$$

Assuming a real conductivity (i.e., no chargeability), this becomes

$$L = \frac{2c\epsilon_r'}{\omega\epsilon_r'' + \sigma'/\epsilon_0} \tag{15}$$

with ϵ_r' and ϵ_r'' the real and imaginary parts of the relative permittivity.

4 Microscopically motivated medium models

Several models have been proposed in the literature to explain experimentally measured functions $\chi(\omega)$ (or $\epsilon(\omega)$) and $\sigma(\omega)$. Some of the popular models used in classical (low frequency) EM methods such as Cole-Cole [2, 3] are however unsuitable for a time-domain analysis or simulation because the functions are non-polynomial so have no corresponding local PDE in the time-space domain. We therefore focus on local models here.

4.1 Polarization models

For the case $\sigma = 0$ the simplest model is the Debye model [4], which postulates

$$\tau\mathbf{P}_t + \mathbf{P} = \epsilon_0 a\mathbf{E} \tag{16}$$

with a some number. This is motivated by considering the electrical force on dipoles immersed in liquid. The damping is caused by molecular collisions which is expressed in the Debye relaxation time τ and the dipole mass is ignored. This results in a susceptibility of

$$\chi = \frac{a}{1 + i\tau\omega}. \tag{17}$$

An extension of the Debye model including inertial effects was proposed in [15] with (16) becoming

$$m\mathbf{P}_{tt} + \tau\mathbf{P}_t + \mathbf{P} = \epsilon_0 a\mathbf{E} \tag{18}$$

with m an effective mass terms which leads to

$$\chi = \frac{a}{1 + i\tau\omega - \omega^2 m}. \tag{19}$$

In (18) we recognize a damped harmonic oscillator driven by the electric field.

A phenomenological model is the Cole-Cole model [2, 3] which postulates

$$\chi = \frac{a}{1 + (i\tau\omega)^s}. \tag{20}$$

with typically $s \approx 0.6$. No time domain model of the form (16) is available, but it is instead thought that many different kind of dipoles obeying something like (16) can behave like the Cole-Cole model. The Dissado-Hill cluster theory [5] further supports this idea. A further refinement is the Havriliak-Negami model [9] which is

$$\chi = \frac{a}{(1 + (i\tau\omega)^s)^p}. \tag{21}$$

These models have no direct physical justification and just fit measured data.

4.2 Conductivity models

The simplest model is the Drude model [7, 8],which predates the Debye model but looks similar. The model assumes free electrons are experiencing a friction force due to collisions. The current contribution ($\sigma\mathbf{E}$ in (5)) is modeled as

$$\tau_l \mathbf{J}_t + \mathbf{J} = b\mathbf{E} \tag{22}$$

with b some number and $\tau_l \approx 10^{-14}$. There is also the Lorentz-Drude model [12, 12, 12], which adds a mass term $m\mathbf{J}_{tt}$ to (22). The Drude model leads to

$$\sigma = \frac{b}{1 + i\omega\tau_l}. \tag{23}$$

These models were first proposed to describe free electrons in metals but can also applied to more complicated conduction mechanisms.

5 The extended Debye model

For our purposes it suffices to consider a plane wave, governed by the 1+1 dimensional Maxwell's equation with a generalized Debye model, which includes a dipole mass term, and a static conductivity takes a simpler form. In this section all vectors live in the xy plane and depend only on z. Time differentiation is indicated with a dot and z differentiation by a prime. The dual of a vector $\mathbf{u}$ is denoted by $\mathbf{u}^*$ with $u_x^* = u_y$ and $u_y^* = -u_x$. Maxwell's equation with a generalized Debye model for polarization become

$$\dot{\mathbf{B}} = \mathbf{E}^{*'} \tag{24a}$$

$$\epsilon_0 \dot{\mathbf{E}} + \dot{\mathbf{P}} = -\frac{1}{\mu_0}\mathbf{B}^{*'} - \sigma\mathbf{E} \tag{24b}$$

$$\lambda^2 \ddot{\mathbf{P}} + \tau\dot{\mathbf{P}} + \mathbf{P} = \epsilon_0 \chi \mathbf{E}. \tag{24c}$$

This decouples into two sets of equation for each polarization mode, say $B = B_y$ and $E = E_x$.

$$\dot{B} = -E' \tag{25a}$$

$$\epsilon_0 \dot{E} + \dot{P} = \frac{1}{\mu_0} B' - \sigma E \tag{25b}$$

$$\lambda^2 \ddot{P} + \tau \dot{P} + P = \epsilon_0 \chi E. \tag{25c}$$

Since we now deal with scalar functions of time and one coordinate let's rename z x. The functions to solve for are $E(x,t)$, $B(x,t)$, and $P(x,t)$ (though we are not directly interested in P). Parameters with a subscript zero are constants, and σ and χ are prescribed functions of x only and have discontinuities. At $t = 0$ $B(x,0)$ and $E(x,0)$ are prescribed functions and $P(x,0) = \dot{P}(x,0) = 0$.

The goal is to stably solve (25) with a scheme that allows $\tau = 0$ and $\lambda = 0$ as a limiting case that is treated in [1].

Let us change notation slightly here and write the PDE system as

$$B_t = -E_x, \tag{26a}$$

$$\epsilon_0 E_t + P_t = -\frac{1}{\mu_0} B_x - \sigma E, \tag{26b}$$

$$\lambda^2 P_{tt} + \tau P_t + P = \epsilon_0 \chi E. \tag{26c}$$

Here the functions $B(x,t)$, $E(x,t)$ and $P(x,t)$ are subject to initial conditions at $t = 0$ (in particular, $P(x,0) = \dot{P}(x,0) \equiv 0$). Assume also boundary conditions on E, say Dirichlet or periodic or absorbing. The subscript t denotes differentiation in time and x likewise in space. The given nonnegative parameter functions $\sigma(x)$ and $\chi(x)$ may be only piecewise continuous in space. The parameters with subscript zero are given nonnegative constants.

A standard procedure for eliminating B is to differentiate (26a) by x and (26b) by t. Then substitution for $B_{xt} = B_{tx}$ yields

$$\epsilon_0 E_{tt} + \sigma E_t + P_{tt} - \frac{1}{\mu_0} E_{xx} = 0. \tag{27}$$

Ignoring P for a moment, note that this PDE changes from hyperbolic to parabolic in the passage $\epsilon_0/\sigma \to 0$.

5.1 Frequency domain analysis

In the frequency domain (27) and (25c) determine the spectral relation between wave number k and angular frequency ω, and hence the propagation velocity and decay rate of monochromatic waves in the stationary case. We obtain

$$\tilde{P} = G \epsilon_0 \chi \tilde{E}. \tag{28a}$$

with

$$G = 1/(1 - \lambda^2\omega^2 + i\tau\omega) \tag{28b}$$

where the tilde denotes the Fourier transform in space and time. Putting everything together we get

$$k^2 = \frac{\omega^2}{c^2}(1 + G\chi - i\frac{\sigma}{\epsilon_0\omega}). \tag{28c}$$

Defining

$$Z = GG* \tag{28d}$$

this becomes

$$k^2 = \frac{\omega^2}{c^2}(1 + (1 - \lambda^2\omega^2)Z\chi - i(\frac{\sigma}{\epsilon_0\omega} + \tau Z\omega\chi)). \tag{28e}$$

We solve (28e) for

$$k = k_1 + ik_2 \tag{28f}$$

and obtain a relative dielectric "constant"

$$\epsilon_r = (k_1 c/\omega)^2 \tag{28g}$$

and skindepth

$$L = 1/k_2. \tag{28h}$$

For the pure Debye model ($\lambda = 0$) after some algebra we arrive at the following explicit formulae.

$$k_1 = \frac{\omega}{c}\sqrt{\frac{\sqrt{\alpha^2 + \beta^2} + \beta}{2}}, \tag{29a}$$

$$k_2 = \frac{\omega}{c}\sqrt{\frac{\sqrt{\alpha^2 + \beta^2} - \beta}{2}}, \tag{29b}$$

where we have defined

$$Q = 1 + \omega^2\tau^2, \tag{29c}$$

$$\alpha = \frac{\omega\tau\chi}{Q} + \frac{\sigma}{\epsilon_0\omega}, \tag{29d}$$

$$\beta = 1 + \chi/Q. \tag{29e}$$

A MATLAB script to plot skindepth (as in for example Fig. 6 in [6]) and effective relative permittivity using this formulae is provided below.


```
c = 299792458;
ep0 = 8.85e-12;
mu = 1.26e-6;
sigs = [.075e-3];
om0 = 2*pi*65e6;
tau = 4e-10;
chi =5.7-1;
f1 = 1e6;
f2 = 100e6;
Lmax = 200;
om = 2*pi*linspace(f1,f2,100000);

N = length(sigs);

for k=1:N
    sig = sigs(k);
    Q = @(om) (1+om.^2.*tau.^2);
    al = @(om) (om.*tau.*chi./Q(om) + (sig/ep0)./om);
    bet = @(om)(1+chi./Q(om));

    k1 = om/c.*sqrt(sqrt(al(om).^2+bet(om).^2)+bet(om))/sqrt(2);
    k2 = om/c.*sqrt(sqrt(al(om).^2+bet(om).^2)-bet(om))/sqrt(2);
    L = 1./k2;

    figure(1)
    if(k==1)
        clf;
    end
    lw = 2;
    f = om/(2*pi)*1e-6;
    semilogx(f,L,'linewidth',lw);
    ylabel 'skindepth'
    xlabel 'f (MHz)'

    axis([f(1) f(end) 0 Lmax]);
    if(k==1)
        hold on;
    end
end
title('Skindepth in Debye model');

for k=1:N
```

```
    sig = sigs(k);
    Q = @(om) (1+om.^2.*tau.^2);
    al = @(om) (om.*tau.*chi./Q(om) + (sig/ep0)./om);
    bet = @(om)(1+chi./Q(om));

    k1 = om/c.*sqrt(sqrt(al(om).^2+bet(om).^2)+bet(om))/sqrt(2);
    k2 = om/c.*sqrt(sqrt(al(om).^2+bet(om).^2)-bet(om))/sqrt(2);
    ep = (k1*c./om).^2;;

    figure(2)
    if(k==1)
        clf;
    end
    lw = 2;
    f = om/(2*pi)*1e-6;
    semilogx(f,ep,'linewidth',lw);
    ylabel 'DC'
    xlabel 'f (MHz)'
    if(k==1)
        hold on;
    end
end
title('Relative permittivity in Debye model');
```

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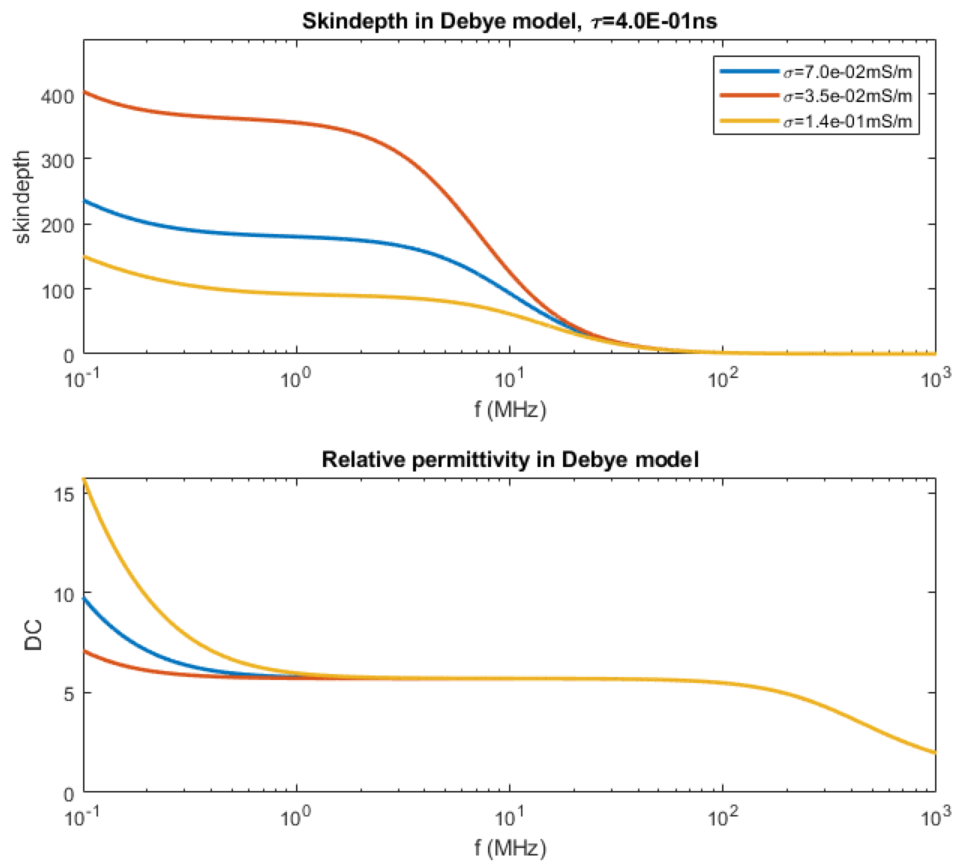

Figure 3: Skindepth (in meters) and relative permittivity (DC) as a function of frequency for some conductivities. $\sigma = .07mS/m$ and $\tau = 0.4ns$ are the parameters measured in [6]. Note the almost horizontal region in the bottom plot over the range $1 - 100MHz$, indicating that velocity is mostly frequency independent in that range, a necessary condition for the transmission of wide-band wave packets, to avoid dispersion.